TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 172, October 1972

EXTREME INVARIANT MEANS WITHOUT MINIMAL SUPPORT([1])

BY

LONNIE FAIRCHILD

ABSTRACT. Let S be a left amenable semigroup. We show that if S has a subset satisfying a certain condition, then there is an extreme left invariant mean on S whose support is not a minimal closed invariant subset of βS. Then we show that all infinite solvable groups and countably infinite locally finite groups have such subsets.

1. **Introduction.** If S is a left amenable semigroup with the discrete topology and βS its Stone-Čech compactification, then S acts on βS by the continuous extension to βS of left translations by elements of S. There is a one-to-one correspondence between the ergodic measures on βS and the extreme points of the set of left invariant means on $m(S)$. It is easy to show that every minimal closed invariant subset of βS is the support of at least one ergodic measure. Wilde, in [8], gives a condition for the support of every ergodic measure on βS to be a minimal closed invariant set, and Chou has proved that there are ergodic measures on βN without minimal support [1].

In §3 we show that a necessary and sufficient condition for the existence of such nonminimally supported ergodic measures is the following: S has a subset A such that $\phi(\chi_A) > 0$ for some left invariant mean ϕ on $m(S)$, but $\psi(\chi_{K^{-1}A}) < 1$ for every left invariant mean ψ and every finite subset K of S. In §4 we prove that every countably infinite locally finite group contains such a subset and that every infinite solvable group has a subset A with the following stronger property: $\phi(\chi_A) > 0$ for some left invariant mean ϕ on $m(S)$, and for each positive integer k there is a finite set $S_k \subset S$ such that $S_k s \not\subset K^{-1}A$ for every $s \in S$ and every subset K of S with $|K| \leq k$.

2. **Preliminaries.** We will follow Day [2] for notation and terminology on invariant means. Let S be a semigroup, $m(S)$ the Banach space of bounded real functions on S with the sup norm, $m(S)^*$ the dual of $m(S)$, and $l_1(S)$ the space of real functions θ on S such that $\|\theta\|_1 = \Sigma_{s\in S} |\theta(s)|$ is finite. Let Q denote the isometry from $l_1(S)$ into $m(S)^*$ given by $Q\theta(f) = \Sigma_{s\in S} f(s)\theta(s)$ for $\theta \in l_1(S)$ and $f \in m(S)$. If we identify each element s of S with the characteristic function $\chi_{\{s\}}$ in $l_1(S)$, we get $Qs(f) = f(s)$ for all $f \in m(S)^*$. For s in S, the operator l_s from $m(S)$ into $m(S)$ is

Presented to the Society, April 4, 1971; received by the editors June 30, 1971.
AMS (MOS) subject classifications (1970). Primary 43A07.
Key words and phrases. Amenable semigroup, extreme point, Stone-Čech compactification, minimal invariant set.
([1]) This paper contains results from the author's Ph. D. thesis (University of Illinois, June 1970). The author wishes to thank her advisor, Professor M. M. Day, for his help.

defined by $l_s f(t) = f(st)$ for all t in S and f in $m(S)$. Let l_s^* be the adjoint of l_s.

$M(S)$ will denote the set of means on $m(S)$, and $Ml(S)$ the set of left invariant means. We will regard βS as the set of multiplicative means with the w^* topology. If U is a subset of βS, U^- will denote the w^* closure of U. If $A \subset S$, then $(QA)^-$ is open in βS and all the open and closed subsets of βS are of this form. βS is extremally disconnected and the open and closed sets form a base for the topology. For other properties of βS, see Gillman and Jerison [3].

The Arens multiplication $\odot$ on $m(S)^*$ is defined as follows: If $\mu, \nu \in m(S)^*$ and $f \in m(S)$, let $\mu \odot \nu(f) = \mu(\nu \odot f)$ where $\nu \odot f(s) = \nu(l_s f)$ for all $s \in S$. This multiplication makes $m(S)^*$ into a Banach algebra. We will use the following properties of $\odot$ (see Day [2]):

(1) $M(S)$ and βS are each closed under $\odot$. (2) If $s \in S$ and $\mu \in m(S)^*$, then $Qs \odot \mu = l_s^* \mu$. (3) $Q(st) = Qs \odot Qt$ for all $s, t \in S$. (4) If $\nu \in m(S)^*$ is fixed, then the mapping $\mu \to \mu \odot \nu$ is w^*-w^* continuous. If $\theta \in l_1(S)$ is fixed, the mapping $\mu \to Q\theta \odot \mu$ is w^*-w^* continuous. (5) If $\mu \in Ml(S)$ and $\nu \in M(S)$ then $\mu \odot \nu \in Ml(S)$ and $\nu \odot \mu = \mu$.

Let $C(\beta S)$ be the space of real-valued continuous functions on βS. Let T be the isometric algebra isomorphism $C(\beta S)$ onto $m(S)$ given by $Tf(s) = f(Qs)$ for all $s \in S$ and $f \in m(S)$. Define P from $m(S)^*$ onto $C(\beta S)^*$ to be the adjoint of T. By the Riesz representation theorem we can regard $C(\beta S)^*$ as the space of regular Borel measures on βS.

If $K \subset S$ and $U \subset \beta S$, let $QK^{-1} \odot U = \{\mu \in \beta S: Qs \odot \mu \in U$, for some $s \in K\}$. If $A \subset S$, let $K^{-1}A = \{t \in S: st \in A$ for some $s \in K\}$. If $\mu \in m(S)^*$, then $\mu \in Ml(S)$ iff $P\mu$ is a probability measure on βS such that $P\mu(Qs^{-1} \odot U) = P\mu(U)$ for each $s \in S$ and each Baire subset U of βS. $P\mu$ is called *invariant* if μ is left invariant. If S is an infinite right cancellation semigroup and $\mu \in Ml(S)$ then $P\mu(QS) = 0$.

An invariant probability measure λ on βS is called *ergodic* if $\lambda(A) = 0$ or $\lambda(A) = 1$ for all Baire sets A which satisfy $\lambda(A \,\Delta (Qs^{-1} \odot A)) = 0$ for each $s \in S$. A left invariant mean μ on $m(S)$ is an extreme point of the convex set $Ml(S)$ iff $P\mu$ is ergodic. (See Phelps [7, p. 81].)

A nonempty subset U of βS is said to be *invariant* if $Qs \odot U \subset U$ for all $s \in S$. In this case we have $\omega \odot U \subset U$ for all $\omega \in \beta S$. U is called *minimal* if it is closed and invariant and minimal with respect to this property. If $\omega \in \beta S$, let $o(\omega) = \{Qs \odot \omega: s \in S\}$. Then $o(\omega)^-$ is a closed invariant set.

If S is left amenable and F is a closed invariant set, then for each $\omega \in F$ and $\mu \in Ml(S)$, $P(\mu \odot \omega)$ is an invariant measure with support contained in F. The following proposition was proved by Wilde and Witz [9, p. 583] for the case where S has two-sided cancellation.

2.1 **Proposition.** *If S is a semigroup and $\mu \in Ml(S)$ then the support of $P\mu$ is an invariant set.*

Proof. Suppose $\omega \in \operatorname{supp} P\mu$ and $s \in S$. Let QA^- be a basic open neighborhood of $Qs \odot \omega$. Then $Qs^{-1} \odot QA^-$ is open and contains ω. Thus, $0 < P\mu(Qs^{-1} \odot QA^-) = P\mu(QA^-)$. Since QA^- was arbitrary, we get $Qs \odot \omega \in \operatorname{supp} P\mu$.

We say that an element ω of βS is a *left almost periodic* point provided that for every neighborhood U of ω there is a subset A of S satisfying: (1) there exists a finite subset K of S such that $S = K^{-1}A$, and (2) $QA \odot \omega \subset U$.

3. **Minimal sets and supports of invariant measures.** In this section we consider the relations among the set A^S of almost periodic points in βS, the set B^S of elements of βS which belong to some minimal set, and the set K^S of points which are in the support of some invariant measure. We prove that $A^S = B^S$ and $A^S \subset K^S$ for any left amenable semigroup S. The main result is Proposition 3.4 in which we show that if S has a C-subset, then $K^S \setminus A^S \neq \emptyset$.

From now on S is always assumed to be a left amenable semigroup. It is proved in Gottschalk and Hedlund [4, p. 32] that if S is a group then $A^S \subset B^S$. We extend this by the following proposition.

3.1 **Proposition.** *If S is any left amenable semigroup, then $A^S = B^S$.*

Proof. First, let T be a minimal subset of βS and assume $\omega \in T$. Since T is minimal, $T = o(\omega)^-$. Let U be an open neighborhood of ω and suppose there exists an element ν in $T \setminus (QS^{-1} \odot U)$. If $s \in S$, then $Qs \odot \nu$ is in T since T is invariant. If $Qs \odot \nu$ is in $QS^{-1} \odot U$, there exists $t \in S$ such that $Q(ts) \odot \nu = Qt \odot Qs \odot \nu \in U$. Then we get $\nu \in Q(ts)^{-1} \odot U$, which contradicts our assumption that ν is not in $QS^{-1} \odot U$. Hence we must have $Qs \odot \nu \in T \setminus (QS^{-1} \odot U)$, so $T \setminus (QS^{-1} \odot U)$ is a closed invariant set. Since T is minimal, this implies that $T \cap (QS^{-1} \odot U) = \emptyset$. But then ω is not in $QS^{-1} \odot U$ and $o(\omega) \cap U = \emptyset$. This is a contradiction since U is open and $\omega \in T = o(\omega)^-$. Therefore, $T \setminus (QS^{-1} \odot U) = \emptyset$ and $T \subset QS^{-1} \odot U$. Since T is compact, there is a finite subset K of S such that $T \subset QK^{-1} \odot U$. Let $A = \{s \in S \colon Qs \odot \omega \in U\}$. If $s \in S$, there is a $k \in K$ such that $Q(ks) \odot \omega = Qk \odot Qs \odot \omega \in U$, so $ks \in A$. Hence $S = K^{-1}A$ and $\omega \in A^S$.

Now, assume $\omega \in A^S$ and let T be a minimal subset of $o(\omega)^-$. Suppose ω is not in T. Let U be an open neighborhood of ω such that $U^- \cap T = \emptyset$. S has a subset A and a finite subset K such that $QA \odot \omega \subset U$ and $S = K^{-1}A$. Since $k \in K$, $s \in S$, and $ks \in A$ implies that $Qs \odot \omega \in Qk^{-1} \odot QA \odot \omega \subset Qk^{-1} \odot U$, we have $o(\omega) \subset QK^{-1} \odot U$. Hence $o(\omega)^- \subset (QK^{-1} \odot U)^- \subset QK^{-1} \odot U^-$. If $k \in K$ and $(Qk^{-1} \odot U^-) \cap T \neq \emptyset$, for some $\mu \in \beta S$ we would have $Qk \odot \mu \in U^- \cap (Qk \odot T) \subset U^- \cap T = \emptyset$, which is impossible. This gives $T \subset (o(\omega)^- \cap T) \subset (QK^{-1} \odot U^-) \cap T = \emptyset$, which contradicts $T \neq \emptyset$. Therefore, $\omega \in T$.

3.2 **Corollary.** $A^S \subset K^S$.

Proof. If T is minimal and contains the support of $P\mu$, then $T = \operatorname{supp}(P\mu)$.

If $A \subset S$, we define $\bar{d}(A)$ to be $\sup\{\phi(\chi_A): \phi \in Ml(S)\}$. Mitchell [6, p. 253] proved that for each f in $m(S)$, the set $\{\phi(f): \phi \in Ml(S)\}$ is a closed interval. Hence, $\bar{d}(A) = 1$ iff $\phi(\chi_A) = 1$ for some $\phi \in Ml(S)$.

If $A \subset S$, we call A a *C-subset* of S provided that

(a) $\bar{d}(A) > 0$, and

(b) $\bar{d}(K^{-1}A) < 1$ for every finite subset K of S.

3.3 **Lemma.** *If S is a semigroup, $A \subset S$, and $s \in S$, then $Qs^{-1} \odot QA^- = (Qs^{-1} \odot QA)^- = Q(s^{-1}A)^-$.*

Proof. Clearly, $Q(s^{-1}A)^- \subset (QS^{-1} \odot QA)^- \subset Qs^{-1} \odot QA^-$. Suppose $\omega \in Qs^{-1} \odot QA^-$. Then $Qs \odot \omega \in QA^-$. Let (Qt_α) be a net in QS which converges to ω. Then $\lim Q(st_\alpha) = \lim (Qs \odot Qt_\alpha) = Qs \odot \omega \in QA^-$. Since QA^- is open, there exists α_0 such that $\alpha > \alpha_0$ implies $Q(st_\alpha) \in QA^-$. But $QS \cap QA^- = QA$, so $st_\alpha \in A$ for $\alpha > \alpha_0$ and thus $t_\alpha \in s^{-1}A$. Hence $\omega = \lim Qt_\alpha \in Q(s^{-1}A)^-$, which shows that $Qs^{-1} \odot QA^- = Q(s^{-1}A)^-$.

3.4 **Proposition.** *Suppose S contains a C-subset A. Then $QA^- \cap (A^S)^- = \emptyset$ and $K^S \cap QA^- \neq \emptyset$ (and thus $A^S \subsetneq K^S$).*

Proof. Assume there exists $\omega \in QA^- \cap A^S$. Then, since QA^- is open, S has a subset B and a finite subset K such that $S = K^{-1}B$ and $QB \odot \omega \subset QA^-$. Thus, $o(\omega) = Q(K^{-1}B) \odot \omega \subset QK^{-1} \odot QB \odot \omega \subset QK^{-1} \odot QA^- = Q(K^{-1}A)^-$, by Lemma 3.3. Therefore, $o(\omega)^- \subset Q(K^{-1}A)^-$. Since $o(\omega)^-$ is a closed invariant set, there exists $\phi \in Ml(S)$ such that $\operatorname{supp}(P\phi) \subset Q(K^{-1}A)^-$. Then $\phi(\chi_{K^{-1}A}) = 1$, which contradicts the definition of C-subset. Thus $QA^- \cap A^S = \emptyset$. Since QA^- is open, $QA^- \cap (A^S)^- = \emptyset$. But, since $\bar{d}(A) > 0$, $QA^- \cap K^S \neq \emptyset$.

3.5 **Corollary.** *Suppose S has a C-subset A. Then there is an extreme point ϕ of $Ml(S)$ such that the support of $P\phi$ is not a minimal set.*

Proof. There exists $\psi \in Ml(S)$ with $\psi(\chi_A) > 0$, so there is an extreme point ϕ of $Ml(S)$ with $\phi(\chi_A) > 0$. Then $QA^- \cap \operatorname{supp}(P\phi) \neq \emptyset$. But $QA^- \cap A^S = \emptyset$, so by Proposition 3.1, $\operatorname{supp}(P\phi)$ is not a minimal set.

Chou implicitly proves 3.4 and 3.5 in [1] for the case where $S = N$. Since $QA^- \cap (A^S)^- = \emptyset$ in Corollary 3.5, we have actually shown that the support of $P\phi$ properly contains the closure of the set $\bigcup \{T \subset \operatorname{supp}(P\phi): T \text{ is a minimal set}\}$. Thus ϕ is not in the w^* closed convex hull of the set $\{\mu \in Ml(S): \operatorname{supp}(P\mu) \text{ is a minimal set}\}$.

3.6 **Proposition.** *Suppose S does not have any C-subsets. Then $K^S \subset (A^S)^-$.*

Proof. Let ω be in K^S and let U be an open neighborhood of ω. There is a subset A of S such that $\omega \in QA^- \subset U$. There is a left invariant mean ϕ with $\phi(\chi_A) > 0$. Since A is not a C-subset of S there is a finite subset K of S such that $\bar{d}(K^{-1}A) = 1$. Thus there exists $\psi \in Ml(S)$ such that $\text{supp}(P\psi) \subset Q(K^{-1}A)^-$. Hence there is a minimal set $T \subset Q(K^{-1}A)^- \subset QK^{-1} \odot QA^-$. Choose $\omega_0 \in T$ and $k \in K$ so that $QK \odot \omega_0 \in QA^- \subset U$. Then $Qk \odot \omega_0 \in T \cap U$, so $U \cap A^S \neq \emptyset$ by Proposition 3.1. Since U was arbitrary, we get $\omega \in A^S$.

4. Semigroups with C-subsets and C'-subsets. In this section we prove the existence of C-subsets for certain semigroups.

Z will denote the additive group of integers, and N the positive integers. If $x, y \in Z$, let $[x, y] = \{a \in Z: x \leq a \leq y\}$. If $k \in N$, a *k-chain* is a subset A of Z such that $|c - d| \leq k$ whenever c and d are two consecutive elements of A. A k-chain A has *length n* if $|A| = n$. If B is a set, $|B|$ denotes the cardinality of B.

A subset A of a semigroup S is called *left thick* if for every finite subset K of S, there exists $s_K \in S$ such that $Ks_K \subset A$. We need the following theorem.

4.1 **Theorem** (Mitchell [6, p. 257]). *If S is a left amenable semigroup and $A \subset S$, then $\bar{d}(A) = 1$ iff A is left thick.*

A subset A of S is said to be a *C'-subset* of S if

(1) $\bar{d}(A) \geq 0$, and

(2) for each $k \in N$, there exists a finite subset S_k of S such that for all $s' \in S$ and $K \subset S$ with $|K| \leq k$ we have $S_k s' \not\subset K^{-1}A$.

Condition (2) implies that for each finite set K, $K^{-1}A$ is not left thick, so by Theorem 4.1, every C'-subset of S is a C-subset.

Chou [1] has constructed a C-subset of N (or Z). We will generalize this construction to show that Z (or N) has a large number of C'-subsets.

4.3 **Theorem.** *If A is a left thick subset of Z (or N), then A contains a C'-subset D of Z (or N).*

Proof. We will prove the proposition for Z. Either $A \cap N$ or $A \cap (-N)$ is left thick. We will assume $A \cap N$ is left thick; a symmetrical argument works in the other case.

By the definition of left thickness, there exists b_1 in A such that $b_2 = b_1 + 1 \in A$. Let $B_1 = \{b_1, b_2\}$. Assume $B_1, B_2, \cdots, B_k$ and $b_1, \cdots, b_p$, where $p = \sum_{i=1}^k 2^i$, are defined. There exists $b_{p+1} \in A$ such that $b_{p+1} > b_p$ and $B_{k+1} = [b_{p+1}, b_{p+1} + 2^{k-1} - 1] \subset A$. Let $b_{p+j+1} = b_{p+1} + j$ for $j = 0, 1, \cdots, 2^{k+1} - 1$. Then $B = \bigcup \{B_k: k \in N\}$ is a left thick subset of A.

Let $A_1 = \{b_1, b_2, b_3\}$. Assume we have constructed $A_1, \cdots, A_n$. Let $a(n) = |A_n|$ and let $d(n) = |\{b_i \in B: b_i \leq \sup A_n, b_i \in A_n\}|$. Define $A_{n+1} = A_n \cup \{b_{a(n)+d(n)+n+j}: b_j \in A_n\} \cup \{b_{2a(n)+2d(n)+2n+j}: b_j \in A_n\}$. Let $D = \bigcup \{A_i: i \in N\}$. If we let $J_n = \{i: b_i \in A_n\}$ for each n in N, then we get $J_{n+1} = J_n \cup (J_n + n + \sup J_n) \cup (J_n + 2n + 2 \sup J_n)$.

For n in N, define ϕ_n in $M(Z)$ by $\phi_n(f) = (1/n) \sum_{i=1}^n f(b_i)$ for all f in $m(Z)$.

We will show that the sequence (ϕ_n) is w^* convergent to left invariance. For each n, let $k(n)$ be the integer which satisfies $b_n \in B_{k(n)}$. Then $n \geq 2^{k(n)} - 1$, and

$$|\phi_n(f) - \phi_n(l_1 f)| = \frac{1}{n}\left|\sum_{i=1}^{n} f(b_i) - f(b_i + 1)\right|$$

$$= \frac{1}{n}\left|\sum_{j=1}^{k(n)} \sum_{\{i \leq n:\, b_i \in B_j\}} f(b_i) - f(b_i + 1)\right|$$

$$\leq 2\|f\| k(n)/n \leq 2\|f\| k(n)/2^{k(n)-1}.$$

Thus $\lim |\phi_n(f) - \phi_n(l_1 f)| = 0$. Let $\psi_n = \phi_{a(n)+d(n)}$ for each n in N and let ψ be a w^* limit point of the sequence (ψ_n). Then $\psi \in Ml(Z)$. We have

$$\psi_n(\chi_D) = |D \cap \{b_1, b_2, \cdots, b_{a(n)+d(n)}\}|/(a(n) + d(n)) = a(n)/(a(n) + d(n)).$$

We have $a(n+1) = 3a(n)$ for n in N, and $a(1) = 3$. Hence $a(n) = 3^n$ for each n. Also $d(n+1) = 3d(n) + 2n$ for n in N and $d(1) = 0$. Hence $d(n) = 1 - n + (3^n - 3)/2$. This gives $\psi_n(\chi_D) = 3^n/(3^n + 1 - n + (3^n - 3)/2) \geq 2/3$, so $\bar{d}(D) \geq \psi(\chi_D) \geq 2/3$.

Note that if $k \in N$, every k-chain in D has length $\leq a(k)$. Also, if E is a k-chain of length n then $(\sup E - \inf E) \leq (n-1)k$, and if $y - x \geq nk + k$ then $[x, y]$ contains an interval of the form $[p, p + k - 1]$ which does not meet E. We will use the following lemma to show that D satisfies condition (2) for a C'-subset.

4.4 **Lemma.** *Assume $k \in N$. Then for each $j \in N$ there exists $p_k(j) \in N$ such that, for any finite subset K of Z with $|K| \leq k$ and for any $z \in Z$, there exists $z_0 \in [z, z + p_k(j) - j]$ with $[z_0, z_0 + j - 1] \cap (-K + D) = \emptyset$.*

Proof. The proof is by induction on k. Take $p_1(j) = ja(j) + j$ for each j in N. Since D contains no j-chain of length $\geq a(j)$, neither does $-s + A_0$ for any s in Z. Then, if $z \in Z$, the set $[z, z + p_1(j) - 1] \cap (-s + A_0)$ is not a j-chain. Thus, there exists $z_0 \in [z, z + p_1(j) - j]$ such that $[z_0, z_0 + j - 1] \cap (-s + A_0) = \emptyset$, so the $p_1(j)$'s satisfy the lemma.

Now, assume there exist $p_n(j)$'s satisfying the lemma. Let $p_{n+1}(j) = p_n(p_1(j))$ for each $j \in N$. Suppose $K \subset Z$ and $1 \leq |K| \leq n + 1$. We can write $K = K_n \cup \{a\}$ where $|K_n| \leq n$. Then $-K + D = (-K_n + D) \cup (-a + D)$. Suppose $z \in Z$. There exists $z_0 \in [z, z + p_n(p_1(j)) - p_1(j)]$ such that $[z_0, z_0 + p_1(j) - 1] \cap (-K_n + A_0) = \emptyset$. There exists $z_0' \in [z_0, z_0 + p_1(j) - j]$ such that $[z_0', z_0' + j - 1] \cap (-a + A_0) = \emptyset$. Then z_0' is in $[z, z + p_{n-1}(j) - j]$ and $[z_0', z_0' + j - 1] \cap (-K_n + A_0) = \emptyset$. Hence, $[z_0', z_0' + j - 1] \cap (-K + A_0) = \emptyset$ and the $p_{n+1}(j)$'s satisfy the lemma.

To complete the proof of Theorem 4.3, let $S_k = [0, p_k(1) - 1]$ for each $k \in N$. Then if $s \in Z$, we have $S_k + s = [s, s + p_k(1) - 1]$. Suppose $K \subset Z$ and $|K| \leq k$. Then, by the lemma, there exists $z \in [s, s + p_k(1) - 1]$ such that $z \notin (-K + D)$. Thus $S_k + s \not\subset -K + A_0$ for all $s \in Z$, and D is a C'-subset of Z.

A *locally finite* group is a group in which every finite subset generates a finite subgroup. Note that the countable locally finite groups are precisely those groups which can be obtained as an increasing union of finite subgroups.

4.5 **Theorem.** *Let G be an infinite group such that $G = \bigcup\{G_i: i \in N\}$ where each G_i is a finite subgroup of G and each $G_i \subset G_{i+1}$. Then if A is a left thick subset of G, A contains a C-subset D of G. Moreover, if each G_i is normal in G, then A contains a C'-subset of G.*

Proof. We can assume that $[G_{i+1}: G_i] \geq 2^{i+1}$. Let $H_1 = G_1$. There exists $d_{1,1} \in G$ such that $d_{1,1} \notin H_1$ and $H_1 d_{1,1} \subset A$. Choose G_q so that $H_1 \cup H_1 d_{1,1} \subset G_q$ and let $H_2 = G_q$. Suppose we have defined $H_1, \cdots, H_n$ and $d_{1,1}, \cdots, d_{n-1,1}$ where $n \geq 2$. There exists $d_{n,1} \in G$ such that $d_{n,1} \notin H_n$ and $H_n d_{n,1} \subset A$. Choose G_p such that $H_n \cup H_n d_{n,1} \subset G_p$ and let $H_{n+1} = G_p$. In this way we get a sequence (H_i) of subgroups of G such that $G = \bigcup\{H_i: i \in N\}$, $H_i \cup H_i d_{i,1} \subset H_{i+1}$, $d_{i,1} \notin H_i$, and $[H_{i+1}: H_i] \geq 2^{i+1}$. For each i in N let $d_{i,0} = e$ and let $\{d_{i,0}, d_{i,1}, \cdots, d_{i,m(i)}\}$ be a right transversal of H_i in H_{i+1}, where $d_{i,1}$ is as above. If $i, j \in N$ and $j > i$, define

$$T_{i,j} = \{d_{i,p(i)} d_{i+1,p(i+1)} \cdots d_{j-1,p(j-1)}: 0 \leq p(r) \leq m(r) \text{ for } i \leq r \leq j-1\}$$

and let $T_{i,i} = \{e\}$. Let $T_i = \bigcup\{T_{i,j}: j \geq i\}$. Then $T_{i,j}$ is a right transversal of H_i in H_j and T_i is a right transversal of H_i in G. Also, $T_{i,j} T_{j,k} = T_{i,k}$ if $i \leq j \leq k$. If $1 \leq i < k$, define $B_{k,i} = \bigcup\{H_i d_{i,1} t: t \in T_{i+1,k}\}$ and let $B_k = \bigcup\{B_{k,i}: 1 \leq i < k\}$. Let $D = \bigcup\{(H_k \setminus B_k) d_{k,1}: k \geq 2\}$.

For $j \in N$, define $\phi_j \in M(G)$ by $\phi_j(f) = |H_j|^{-1} \sum_{g \in H_j} f(g d_{j,1})$ for all $f \in m(G)$. Then if $s \in H_j$ and $f \in m(G)$ we have

$$\phi_j(f) - \phi_j(l_s f) = |H_j|^{-1}\left(\sum_{g \in H_j} f(g d_{j,1}) - \sum_{g \in H_j} f(s g d_{j,1})\right) = 0,$$

so the sequence (ϕ_j) is w^* convergent to left invariance. Let ϕ be a w^* limit point of (ϕ_j). If $n \geq 2$,

$$\begin{aligned} |B_n| &\leq \sum_{i=1}^{n-1} |B_{n,i}| \\ &= \sum_{i=1}^{n-1} |H_i|\,|T_{i+1,n}| = \sum_{i=1}^{n-1} |H_i|\,|H_n|/|H_{i+1}| \\ &\leq |H_n| \sum_{i=1}^{n-1} 2^{-(i+1)} \leq |H_n|/2. \end{aligned}$$

Hence, $\phi_n(\chi_{G \setminus D}) = |H_n|^{-1} |(G \setminus D) \cap H_n d_{n,1}| = |H_n|^{-1} |B_n| \leq 1/2$. Thus $\bar{d}(D) \geq \phi(\chi_D) \geq 1/2 > 0$.

Now, let K be a finite subset of G and choose m so that $K^{-1} \subset H_m$. We will show that no right translate of H_{m+1} is contained in $H_m D$. We have $H_m D = \bigcup\{H_m(H_i \setminus D_i)d_{i,1}: i \geq 2\} \subset H_m \cup H_m d_{m,1} \cup \bigcup\{(H_i \setminus H_m B_i)d_{i,1}: i \geq m+1\}$. Suppose there exists $d \in T_{m+1}$ such that $H_{m+1}d \subset H_m D$. We have $H_m \cup H_m d_{m,1} \subset H_{m+1}$ and $H_{m+1} \cap \bigcup\{(H_i \setminus H_m B_i)d_{i,1}: i \geq m+1\} = \emptyset$. We cannot have $H_{m+1}d \subset H_m \cup H_m d_{m,1}$ since $[H_{m+1}:H_m] \geq 2^{m+1}$. Hence we get $H_{m+1}d \subset \bigcup\{(H_i \setminus H_m B_i)d_{i,1}: i \geq m+1\}$. Since the $H_i d_{i,1}$'s are pairwise disjoint and $H_{m+1} \subset H_i$ for each $i \geq m+1$ this implies that there exists $r \geq m+1$ such that $H_{m+1}d \subset (H_r \setminus H_m B_r)d_{r,1}$. Then we write $H_{m+1}d = H_{m+1}d' d_{r,1}$ where $d' \in T_{m+1,r}$. We get $H_{m+1}d' \cap H_m B_r = \emptyset$. But $H_m d_{m,1} d' \subset H_{m+1}d'$ and $H_m d_{m,1} d' \subset B_r$ by the definition of B_r. Hence $H_m d_{m,1} d' = H_m H_m d_{m,1} d' \subset H_m B_r$, which is a contradiction. Therefore $H_m D$ is not left thick, so by Theorem 4.1, $\bar{d}(K^{-1}D) \leq \bar{d}(H_m D) < 1$.

We will use the following lemma to complete the proof.

4.6 **Lemma.** *Under the hypotheses of Theorem 4.5, suppose each G_i is normal in G and let k be in N. Then for each $j \in N$ there exists $q(k, j) \in N$ such that for each subset K of G with $|K| \leq k$ and for each right coset $H_{q(k,j)}g$ of $H_{q(k,j)}$ there is a coset $H_j g'$ of H_j with $H_j g' \subset H_{q(k,j)}g$ and $H_j g' \cap K^{-1}D = \emptyset$.*

Proof. The proof is by induction on k. Let $q(1, j) = j + 2$ for each $j \in N$. We will show first that $H_j d_{j,1} t \cap D = \emptyset$ for all $t \in T_{j+1}$, $t \neq e$. Suppose there exists $t \in T_{j+1}$ such that $t \neq e$ and $H_j d_{j,1} t \cap D \neq \emptyset$. Then there is an integer i such that $H_j d_{j,1} t \cap (H_i \setminus B_i)d_{i,1} \neq \emptyset$. Since $H_j d_{j,1} t \subset G \setminus H_{j+1}$ we must have $i \geq j+1$ and $H_j d_{j,1} t \subset H_i d_{i,1}$. There exists $t' \in T_{j+1,i}$ such that $H_j d_{j,1} t = H_j d_{j,1} t' d_{i,1}$. But by the definition of B_i, $H_j d_{j,1} t \subset B_i$. Hence $H_j d_{j,1} t' d_{i,1} \cap (H_i \setminus B_i)d_{i,1} = \emptyset$, which gives a contradiction and proves our assertion.

Let g be a fixed element of G. Then $g^{-1}H_j d_{j,1} t \cap g^{-1}D = \emptyset$ for all $t \in T_{j+1}$, $t \neq e$. Let $H_{q(1,j)}d$ be any coset of $H_{q(1,j)}$. Choose $d' \in T_{q(1,j)}$ such that $H_{q(1,j)}d' = H_{q(1,j)}gd$. Let $d'' = d_{j+1,1}$ if $d' = e$ and let $d'' = d'$ otherwise. Then $g^{-1}H_{q(1,j)}d' = g^{-1}H_{q(1,j)}gd = H_{q(1,j)}d$, and $H_j(g^{-1}d_{j,1}d'') = g^{-1}H_j d_{j,1} d'' \subset g^{-1}H_{j+1}d'' \subset g^{-1}H_{q(1,j)}d' = H_{q(1,j)}d$. Since $d'' \in T_{j+1}$, we get $H_j(g^{-1}d_{j,1}d'') \cap g^{-1}D = g^{-1}H_j d_{j,1} d'' \cap g^{-1}D = \emptyset$, which proves the lemma for $k = 1$.

Suppose the lemma holds for $k = n$. Let $q(n+1, j) = q(n, q(1, j))$ for each $j \in N$. Assume $K \subset G$ and $1 \leq |K| \leq n+1$. We can write $K = J \cup \{s\}$ where $1 \leq |J| \leq n$. Let $H_{q(n+1,j)}b$ be any coset of $H_{q(n+1,j)}$. By the induction hypothesis, $H_{q(n+1,j)}b = H_{q(n,q(1,j))}b$ contains a coset $H_{q(1,j)}b'$ of $H_{q(1,j)}$ such that $H_{q(1,j)}b' \cap J^{-1}D = \emptyset$. Also $H_{q(1,j)}b'$ contains a coset $H_j b''$ of H_j such that $H_j b'' \cap s^{-1}D = \emptyset$. Therefore, $H_j b'' \cap K^{-1}D = \emptyset$ and $H_j b'' \subset H_{q(1,j)}b' \subset H_{q(n+1,j)}b$, which proves the lemma.

To complete the proof of Theorem 4.5, assume each G_i is normal in G and let $S_k = H_{q(k,1)}$ for each $k \in N$. Suppose $K \subset G$ and $|K| \leq k$. If $s \in G$, then $S_k s = H_{q(k,1)}s$ contains a coset $H_1 d$ of H_1 such that $H_1 d \cap K^{-1}D = \emptyset$. Thus $S_k s \not\subset K^{-1}D$ for all $s \in G$. Therefore D is a C'-subset of G.

Suppose that G is a countably infinite left amenable group and that every left thick subset of G contains a C-subset. Then if A is any left thick subset of G, there is a set $\mathfrak{A}$ consisting of C-subsets of G which are contained in A, such that $|\mathfrak{A}| = c$ and $D \cap E$ is finite for all $D, E \in \mathfrak{A}$ with $D \neq D'$. (See Chou [1, p. 780]. The argument there can be generalized.) Hence if $D, E \in \mathfrak{A}$ and $D \neq E$, we have $(D^- \setminus D) \cap (E^- \setminus E) = \emptyset$. Since $\operatorname{supp} P\mu \subset \beta G \setminus G$ for each $\mu \in Ml(G)$, we have shown that for each left thick subset A of G there is a set M_A of extreme left invariant means on G such that $|M_A| = c$, $\operatorname{supp} P\mu \cap \operatorname{supp} P\nu = \emptyset$ for $\mu, \nu \in M_A$ with $\mu \neq \nu$, and $\operatorname{supp} P\mu$ is not a minimal set if $\mu \in M_A$.

4.7 **Theorem.** *Suppose G is an abelian group and suppose a subgroup H of G has a C'-subset B' (of H). Then there exists a C'-subset of G.*

Proof. Let T be a transversal of H in G such that $e \in T$ and let $B = TB'$. Let ϕ be a left invariant mean on $m(H)$ such that $\phi(\chi_B) > 0$ and let μ be any left invariant mean on $m(G)$. If $f \in m(G)$, define $Pf \in m(G)$ by $Pf(g) = \phi(l_g f|_H)$ for all $g \in G$. Define $\nu \in M(G)$ by $\nu(f) = \mu(Pf)$ for all $f \in m(G)$. Then $\nu \in Ml(G)$. If $h \in H$ and $t \in T$ we have $l_t\chi_B(h) = \chi_B(th) = \chi_{B'}(h)$, so $(l_t\chi_B)|_H = \chi_{B'}$. Hence $P\chi_B(th) = \phi(l_{th}\chi_B|_H) = \phi(l_h\chi_B|_H) = \phi(\chi_{B'})$. Thus $\bar{d}(B) \geq \nu(\chi_B) = \mu(P\chi_B) \geq \phi(\chi_{B'}) > 0$.

Suppose $k \in N$ and let S_k be a subset of H with the property that $S_k h \not\subset J^{-1}B'$ for any h in H and for any subset J of H with $|J| \leq k^2$. Suppose $K \subset G$ and $|K| \leq k$. We can write $K^{-1} = \bigcup \{t_i K_i: 1 \leq i \leq n\}$ where each $K_i \subset H$, $n \leq k$, and $t_1, \cdots, t_n \in T$. Let $K' = \bigcup \{K_i: 1 \leq i \leq n\}$ and let $K_0 = \bigcup \{t_i K': 1 \leq i \leq n\}$. Then $K^{-1} \subset K_0$, $|K'| \leq k$, and $|K_0| \leq k^2$. We will show that $S_k s \not\subset K_0 B$ for all $s \in G$. We have $K_0 B = \bigcup \{t_i K' TB'\} = \bigcup \{t_i K' tB': 1 \leq i \leq n,\ t \in T\}$. For each $i \leq n$ and $t \in T$ let $g_{i,t}$ be the unique element of T such that $t_i K' t \subset g_{i,t} H$ and let $K_{i,t} = g_{i,t}^{-1} t_i K' t \subset H$. Then, for fixed i, the set $\{g_{i,t}: t \in T\}$ is a transversal for H in G. If $d \in T$, let $K_d = \bigcup \{K_{i,t}: g_{i,t} = d\}$. Note that $|K_{i,t}| = |K'| \leq k$, so $|K_d| \leq n|K'| \leq k^2$. We have $K_0 B = \bigcup \{g_{i,t} K_{i,t} B': 1 \leq i \leq n,\ t \in T\} = \bigcup \{dK_d B': d \in T\}$. Also, $S_k tH \subset Hth = tH$ for all $t \in T$ and $h \in H$. Suppose $S_k th \subset K_0 B$ for some $t \in T$ and $h \in H$. Then we would have $tS_k h = S_k th \subset K_0 B \cap tH = tK_t B'$. This gives $S_k h \subset K_t B_0$, which contradicts the definition of S_k. Hence $S_k g \not\subset K_0 B \supset K^{-1} B$ for all $g \in G$, so B is a C' subset of G.

4.8 **Corollary.** *If G is an infinite abelian group then G has a C'-subset and there is an extreme left invariant mean ϕ such that ϕ is not in the w^* closed convex hull of the set $\{\mu \in Ml(G)$: supp $P\mu$ is a minimal set$\}$.*

Proof. G has either a subgroup which is isomorphic to Z or a countable periodic subgroup. Thus, by 4.3 and 4.5, G has a subgroup with a C'-subset. By 4.7, G has a C'-subset and Corollary 3.12 gives the result.

The next theorem can be proved by methods similar to those used in the proof of Theorem 4.7. We omit the proof.

4.9 Theorem. *If S is an infinite abelian cancellation semigroup, then S has a C'-subset.*

4.10 Proposition. *Let G be a left amenable group with a normal subgroup H. Suppose G/H contains a C-subset [C'-subset] B. Then G contains a C-subset [C'-subset].*

Proof. Let T be a transversal of H in G and let $T_0 = \{t \in T: tH \in B\}$. Let ϕ be a left invariant mean on $m(G/H)$ such that $\phi(\chi_B) > 0$ and let ψ be any left invariant mean on $m(H)$. If $f \in m(G)$, $t \in T$, and $h \in H$, then $\psi(l_{th}f|_H) = \psi(l_h l_t f|_H) = \psi(l_t f|_H)$. If $f \in m(G)$, define $Pf \in m(G/H)$ by $Pf(tH) = \psi(l_t f|_H)$; then Pf does not depend on T. If we let $\mu(f) = \phi(Pf)$ for all $f \in m(G)$, then μ is a left invariant mean on $m(G)$. Let $A = T_0 H \subset G$. If $tH \in B$, then $tH \subset A$ so $H \subset t^{-1}A$ and $l_t\chi_A(h) = \chi_{t^{-1}A}(h) = 1$ for all $h \in H$. Thus $P\chi_A(tH) = 1$. If $tH \in B$ then $A \cap tH = \emptyset$, so $t^{-1}A \cap H = \emptyset$ and $l_t\chi_A(h) = 0$ for all $h \in H$. This gives $P\chi_A(tH) = 0$. Thus, $P\chi_A = \chi_B$ and $\bar{d}(A) \geq \mu(\chi_A) = \phi(\chi_B) > 0$.

Let K be a finite subset of G and let i be the canonical map of G onto G/H. Let $J \subset G/H$ be a finite set such that $Jy \not\subset i(K)^{-1}B$ for all $y \in G/H$. Let $V = \{t \in T: tH \in J\}$. Suppose there exists $g \in G$ such that $Vg \subset K^{-1}A$. Then we would have $Ji(g) = i(Vg) \subset i(K^{-1}A) = i(K)^{-1}B$ which is impossible by the choice of J. Hence A is a C-subset of G.

Now assume B is a C'-subset of G/H. If $k \in N$, let S_k be a finite subset of G/H such that $S_k y \subset K^{-1}B$ for all $K \subset G/H$ with $|K| \leq k$ and for all $y \in G/H$. Let $V_k = \{t \in T: tH \in S_k\}$. Then if $J \subset G$, $|J| \leq k$, and $g \in G$, the assertion $V_k g \subset J^{-1}A$ would imply that $S_k i(g) \subset i(J)^{-1}B$, which contradicts the definition of S_k. Therefore A is a C'-subset of G.

4.11 Proposition. *Let G be a left amenable group and let H be a subgroup of finite index in G. Suppose A is a C-subset [C'-subset] of H. Then A is a C-subset [C'-subset] of G.*

Proof. Let ϕ be a left invariant mean on $m(H)$ such that $\phi(\chi_A) > 0$. If $f \in m(G)$, define $Pf \in m(G)$ by $Pf(g) = \phi(l_g f|_H)$ for all $g \in G$. Choose $\psi \in Ml(G)$ and define $\mu \in M(G)$ by $\mu(f) = \psi(Pf)$ for all $f \in m(G)$. Then $\mu \in Ml(G)$. If $h \in H$, then $P\chi_A(h) = \phi(l_h\chi_A|_H) = \phi(\chi_A)$. If $g \in G \setminus H$, then $P\chi_A(g) = \phi(l_g\chi_A|_H) = \phi(\chi_{g^{-1}A \cap H}) = 0$. Thus $P\chi_A = \phi(\chi_A)\chi_H$. Also, $\mu(\chi_H) = [G:H]^{-1}$. Thus $\bar{d}(A) \geq \mu(\chi_A) = \phi(\chi_A)\mu(\chi_H) > 0$. It is clear that A satisfies the second condition for a C-subset [C'-subset] of G.

4.12 Theorem. *Every infinite solvable group G contains a C'-subset.*

Proof. Let $H_0 = \{e\}, H_1, \cdots, H_n = G$ be a normal series for G such that H_i/H_{i-1} is abelian for $1 \leq i \leq n$. Since G is infinite we can define $m = \sup\{i: H_i/H_{i-1}$ is infinite$\}$. Then by 4.8, H_m/H_{m-1} has a C'-subset. Thus, by

4.10, H_m contains a C'-subset. If $H_m = G$ we are done. Otherwise, for $0 \leq p \leq n - m - 1$, H_{m+p} contains a C'-subset implies, by 4.11, that H_{m+p+1} contains a C'-subset. By induction, $H_n = G$ contains a C'-subset.

We do not know whether there are any left amenable semigroups S, aside from those which have finite invariant subsets in βS (see [5]), which do not have C-subsets.

REFERENCES

1. C. Chou, *Minimal sets and ergodic measures for* $\beta N \backslash N$, Illinois J. Math. 13 (1969), 777–788. MR 40 #2814.
2. M. M. Day, *Amenable semigroups*, Illinois J. Math. 1 (1957), 509–544. MR 19, 1067.
3. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR 22 #6994.
4. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, R. I., 1955. MR 17, 650.
5. A. Lau, *Topological semigroups with invariant means in the convex hull of the multiplicative means*, Trans. Amer. Math. Soc. 148 (1970), 69–84. MR 41 #1911.
6. T. Mitchell, *Constant functions and left invariant means on semigroups*, Trans. Amer. Math. Soc. 119 (1965), 244–261. MR 33 #1743.
7. R. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N. J., 1966. MR 33 #1690.
8. C. Wilde, *On amenable semigroups and applications of the Stone-Čech compactification*, Thesis, University of Illinois, Urbana, Ill., 1964.
9. C. Wilde and K. Witz, *Invariant means and the Stone-Čech compactification*, Pacific J. Math 21 (1967), 577–586. MR 35 #3423.

DEPARTMENT OF MATHEMATICS, SUNY, COLLEGE AT PLATTSBURGH, PLATTSBURGH, NEW YORK 12901